Sidebar. Adiabatic lapse rate in the atmosphere

We've all experienced the decrease in temperature with elevation on hiking, driving, flying (that -46°C outside the plane on a long-distance flight!). The rate of decrease, or lapse rate, is a very important structure in our weather and our climate. It can be explained in stable atmospheric conditions by the physics of buoyancy of air (hydrostatic support of air parcels) and of the energy shift in air as it expands upon rising. Below is a derivation. Note that it refers to the stable conditions, with no net motion of air upward or downward. There are two common kinds of deviations from this pattern. One is a temperature inversion, with temperature at low elevations being lower than this case. That caps all motion, which traps pollutants, for one (when I arrived in Pasadena, CA in 1966, I could not see the visual boundary between the gray mountains and the gray, smoggy air for 6 weeks). Another deviation is unstable conditions, with air below hotter than the stable lapse rate. Lower air then moves upward with turbulence. This happens very often in the morning as solar radiation is intercepted mainly by the surface, heating it quickly. Air with sufficient humidity can rise a significant distance vertically, cooling adiabatically, which means without significant heat exchange with other air masses (air is a good insulator) or sources of energy (air absorbs solar energy very poorly). The fraction of air as water vapor stays constant until the temperature drops far enough that the vapor pressure of water exceeds the equilibrium vapor pressure at that temperature. Water condenses to mist/fog/clouds. This slows the rate of temperature decrease with elevation because the condensation of water vapor releases the heat of vaporization of water. While the vapor pressure of water in an ascending air mass (before condensation starts) drops in direct ratio with the total air pressure (it's just mutual expansion, at a constant mixing ratio), the equilibrium vapor pressure of water is a steep exponential function of temperature. There will always be an elevation for condensation to start for air with any reasonable humidity. That means we may exclude deserts at times!

After the derivation of the dry adiabatic lapse rate below, I append a program written in Fortran 90. It is intended to cover the wet adiabatic lapse rate, after condensation starts, but it has an unfixed error. If you like physics or can cut-and-paste more or less from Wikipedia, go for it.

A parcel of air, moved uphill (or just up in the air), expands, and therefore does work against the surroundings. This extracts energy. With no source of new energy (poor transfer of heat across large air masses), this means that the internal energy must drop - that is, the temperature must drop.

We can calculate the rate of T drop with elevation, the "adiabatic lapse rate."

We use another principle of physics, that the pressure difference from top to bottom of an air parcel must be big enough to support the air mass from sinking in the gravitational field. This sets the profile of pressure vs. height.

Derivation:

Hydrostatic equilibrium says how P varies with height (and density)

Consider a parcel of air of area A; base is at height y, top at height y+dy

The force on top is (P+dP)A, on bottom is P A; difference is A dP, driving the parcel of air up (dP is negative; P decreases with height)

Gravitational force on the parcel is -mg, where mass $m = \rho M_w V$

with ρ = molar density, M_w = mass per mole (molecular wt.) and V=volume=A dy

Force balance: $A\, dP = - \rho M_w A\, dy\, g$

or $dP/dy = - \rho M_w g$

Now express density ρ in terms of pressure.

First, use the ideal-gas law to express ρ in terms of P and T:

$PV=nRT$; $\rho = n/V = P/(RT)$

How do we relate P and T uniquely? If air is displaced, it settles back in place (it's in equilibrium at all heights)...and the displacement is an adiabatic process - no heat is added or subtracted from air parcel.

$dQ = 0 = dU - dW$ (change in energy content = 0 = change in internal energy - change in work done on surroundings - just conservation of energy)

That is, $dU = dW$ (change in internal energy = loss from work done).

Now express these two changes in terms of changes in pressure and temperature:

$C_v\, dT = -P\, dV = -(RT/V)\, dV$

Take all the T's on one side, all the V's on the other side:

$C_v\, dT/T = -R\, dV/V$

Integrate it from initial state to final state:

$C_v \ln(T/T_0) = -R \ln(V/V_0)$

$\ln(T/T_0) = -(R/C_v) \ln(V/V_0)$

For a gas of diatomic molecules (which most molecules in air are, O_2 and N_2) that can move (translate) and rotate freely, we have $C_v = (5/2) R$ (each "degree of freedom" of motion has a heat capacity of (1/2) R, and there are 5, two rotations and three translational directions).

Exponentiate both sides:

$T/T_0 = (V/V_0)^{-2/5}$, or

$V/V_0 = (T/T_0)^{5/2}$

Now use this to express P variations in terms of T variations:

$P/P_0 = (RT/V) / (RT_0/V_0) = T V_0/(T_0 V) = (T/T_0) (T/T_0)^{5/2} = (T/T_0)^{7/2}$

Invert this, to express T changes in terms of P changes:

$T = T_0 (P/P_0)^{2/7}$

Use this in the density equation

$P/(RT) = P / [RT_0 (P/P_0)^{2/7}$ --> $P^{5/7} P_0^{2/7}/(RT_0)$

Finally, let's integrate the equation of hydrostatic equilibrium. Recall that this was

$dP/dy = \rho M_w g$, and use $\rho = P/(RT)$:

$dP/dy = -P^{5/7} [P_0^{2/7} M_w g]/[RT_0] == -k P^{5/7}$ --> $dP/P^{5/7} = -k\, dy$

Integrating from the ground (y=0) to any height y:

$\int dP\, P^{-5/7} = (P^{2/7}-P_0^{2/7})/(2/7) = -k y$

$P^{2/7}-P_0^{2/7} = -(2/7) [P_0^{2/7} M_w g / (RT_0)] y$

$(P/P_0)^{2/7} -1 = -(2/7)P_0^{-2/7}[P_0^{2/7}g M_w /(RT_0)] y = -(2/7) [g M_w/(RT_0)] y$

$P/P_0 = [1 - (2/7) [g M_w/(RT_0)] y]^{7/2} == [1 - (2/7)ay]^{7/2}$

At small y (near the surface), $[1 - (2/7) a y]^{7/2} \approx 1 - (7/2)(2/7)ay) = 1- ay$

- that is, P falls off linearly with height- about 1% per 80 meters, or e-fold (to 37% of sea-level pressure) in one "scale height" of 8000 m (near top of Mt. Everest), or about 15% at elevation of Las Cruces (1200 m)

Now let's convert from the P profile to the T profile:

$T/T_0 = (P/P_0)^{2/7} = [1-(2/7) ky]^{(7/2)(2/7)} = 1 - (2/7) ky$!

or,

$T = T_0 - (2/7) [g M_w/R] y = T_0 - b y$

The factor in front of y can be evaluated, using $g = 9.8$ m s^{-2}, $M_w = 0.029$ kg mol^{-1} and R = 8.314 J mol^{-1} K^{-1}, to give b = -0.0098 K m^{-1},

that is, 9.8 degrees (Kelvin or Celsius) per 1000 m.

Surprising result: temperature drops linearly with elevation. (It could drop toward absolute zero at 28 km above sea level, except that the absorption of sunlight becomes important as an energy source in the upper atmosphere.

Water vapor absorbs near infrared [not important at high elevations; there's not much water]; CO_2 absorbs thermal infrared everywhere; and ozone absorbs UV, esp. in the stratosphere. Above the stratosphere, the air even gets warmer with elevation.)

Wet adiabatic lapse rate:

We have to consider how condensation of water vapor (as the air cools) releases sensible heat. This slows the rate of temperature decrease with height. There is no analytical solution, but numerically we can find that the rate is about 6 degrees per 1000 m up to the elevation where water begins to freeze. The rate is lower, because the heat of condensation counteracts the cooling.

Consequences:

Colder life zones with rising elevation/ "mirroring" the N-S gradient

Other implications: water vapor condenses out fast with height

-> clouds form as air lifts

-> total water content of air is very limited; equivalent to 2.5 cm depth of liquid water, over the globe. -> water turns over rapidly in the atmosphere (every 9 days)

-> many interesting phenomena: we can track water loss / sources of rain by the T at which they condensed (the isotopic composition is indicative)

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C:\Users\vince\misc>wet_adiabat
Enter P0,T0,RH0pct,y0,dy,dyprint,ymax
88500. 25. 50. 1200. 1. 100. 5000.
eair0= 1584.73
Fortran Pause - Enter command<CR> or <CR> to continue.

y=1300.0; P/P0= 0.9886; P/P0(expon.)= 0.9887; T= 24.02
y=1400.0; P/P0= 0.9772; P/P0(expon.)= 0.9775; T= 23.04
y=1500.0; P/P0= 0.9660; P/P0(expon.)= 0.9665; T= 22.06
y=1600.0; P/P0= 0.9548; P/P0(expon.)= 0.9556; T= 21.08
y=1700.0; P/P0= 0.9438; P/P0(expon.)= 0.9448; T= 20.10
y=1800.0; P/P0= 0.9328; P/P0(expon.)= 0.9341; T= 19.11
y=1900.0; P/P0= 0.9219; P/P0(expon.)= 0.9235; T= 18.13
y=2000.0; P/P0= 0.9112; P/P0(expon.)= 0.9131; T= 17.15
y=2100.0; P/P0= 0.9005; P/P0(expon.)= 0.9028; T= 16.17
y=2200.0; P/P0= 0.8899; P/P0(expon.)= 0.8926; T= 15.19
y=2300.0; P/P0= 0.8793; P/P0(expon.)= 0.8825; T= 14.21
y=2400.0; P/P0= 0.8689; P/P0(expon.)= 0.8725; T= 13.23
y=2500.0; P/P0= 0.8586; P/P0(expon.)= 0.8627; T= 12.25
y=2600.0; P/P0= 0.8483; P/P0(expon.)= 0.8529; T= 11.34
y=2700.0; P/P0= 0.8382; P/P0(expon.)= 0.8433; T= 11.00
y=2800.0; P/P0= 0.8281; P/P0(expon.)= 0.8338; T= 10.67
y=2900.0; P/P0= 0.8182; P/P0(expon.)= 0.8243; T= 10.33
y=3000.0; P/P0= 0.8084; P/P0(expon.)= 0.8150; T=  9.99
y=3100.0; P/P0= 0.7987; P/P0(expon.)= 0.8058; T=  9.65
y=3200.0; P/P0= 0.7891; P/P0(expon.)= 0.7967; T=  9.30
y=3300.0; P/P0= 0.7796; P/P0(expon.)= 0.7877; T=  8.96
y=3400.0; P/P0= 0.7701; P/P0(expon.)= 0.7788; T=  8.61
y=3500.0; P/P0= 0.7608; P/P0(expon.)= 0.7700; T=  8.26
y=3600.0; P/P0= 0.7516; P/P0(expon.)= 0.7613; T=  7.91
y=3700.0; P/P0= 0.7425; P/P0(expon.)= 0.7527; T=  7.56
y=3800.0; P/P0= 0.7335; P/P0(expon.)= 0.7442; T=  7.20
y=3900.0; P/P0= 0.7246; P/P0(expon.)= 0.7358; T=  6.84
y=4000.0; P/P0= 0.7158; P/P0(expon.)= 0.7275; T=  6.48
y=4100.0; P/P0= 0.7071; P/P0(expon.)= 0.7193; T=  6.12
y=4200.0; P/P0= 0.6985; P/P0(expon.)= 0.7111; T=  5.76
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y=4400.0; P/P0= 0.6816; P/P0(expon.)= 0.6951; T=  5.02
y=4500.0; P/P0= 0.6732; P/P0(expon.)= 0.6873; T=  4.65
y=4600.0; P/P0= 0.6650; P/P0(expon.)= 0.6795; T=  4.28
y=4700.0; P/P0= 0.6568; P/P0(expon.)= 0.6718; T=  3.90
y=4800.0; P/P0= 0.6487; P/P0(expon.)= 0.6643; T=  3.52
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y=4900.0; P/P0= 0.6408; P/P0(expon.)= 0.6567; T= 3.14
y=5000.0; P/P0= 0.6329; P/P0(expon.)= 0.6493; T= 2.75